# CHAPTER 3

# ENERGY METHODS

## 3.0. INTRODUCTION

In the previous course, the relations existing between forces and deformations under various loading conditions were considered. The analysis was based on two fundamental concepts, the concept of **stress** and the concept of **strain**. And now third important concept to be introduced is the concept of **strain energy**.

Strain energy of a member is defined as the increase in energy associated with the deformation of the member. The view that the strain energy is equal to the work done by a slowly increasing load applied to the member will be discussed. The strain-energy density of a material will be defined as the strain energy per unit volume and we shall see that it is equal to the area under the stress-strain curve of the material. From the stress-strain diagram of a material we shall also define the modulus of toughness and modulus of resilience of the material.

The elastic strain energy associated with normal stresses will be discussed, first in members under **axial loading** and then in members in **bending**. Later we shall consider the elastic strain energy associated with **shearing stresses** such as in **torsional** loadings of shafts and in **transverse** loadings of beams. Strain energy for a general state of stress will be considered where we shall derive the maximum-distortion-energy criterion for yielding.

## 3.1. STRAIN ENERGY

Consider a rod BC of length L and uniform cross-sectional area A, which is attached at B to a fixed support, and subjected at C to a slowly increasing axial load P (Fig. 2.1). By plotting the magnitude P of the load against the deformation x of the rod, we obtain a certain load-deformation diagram (Fig. 2.2) which is characteristic of the rod BC.

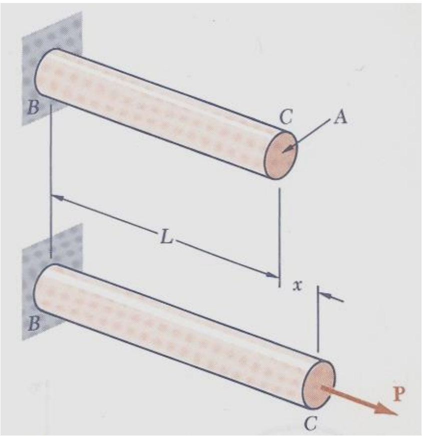


Fig.3.1

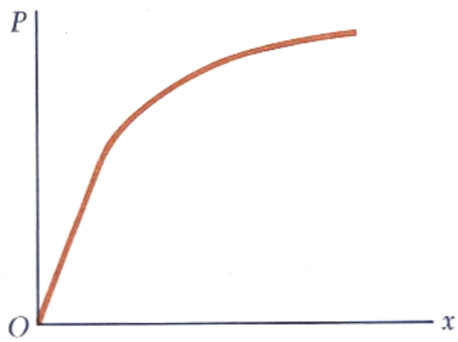


Fig.3.2

Let us now consider the work **dU** done by the load **P** as the rod elongates by a small amount **dx**. This elementary work is equal to the product of the magnitude **P** of the load and of the smallelongation **dx**. We write

**dU = P dx** …(1)

and note that the expression obtained is equal to the element of area of width dx located under the load-deformation diagram (Fig. 3.3). The total work U done by the load as the rod undergoes a deformation $x_1$ is thus

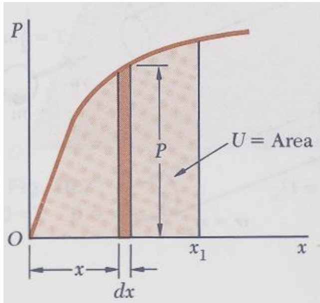

Fig.3.3

$$U = \int_0^{x_1} P\,dx$$

and is equal to the area under the load-deformation diagram between x = 0 and $x = x_1$.

The work done by the load P as it is slowly applied to the rod must result in the increase of some energy associated with the deformation of the rod. This energy is referred to as the **strain energy** of the rod. We have, by definition,

**Strain energy = U =** $\int_0^{x_1} P\,dx$ …(2)

We recall that work and energy should be expressed in units obtained by multiplying units of length by units of force. Thus, if SI metric units are used, work and energy are expressed in N.m; this unit is called a ***joule*** (**J**).

In the case of a linear and elastic deformation, the portion of the load-deformation diagram involved may be represented by a straight lineof equation P = kx (Fig. 2.4). Substituting for P in Eq. (2), we have

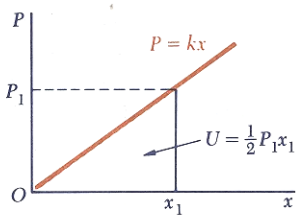

Fig. 3.4

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2$$

or

$$\mathbf{U} = \frac{1}{2}P_1x_1 \ldots(3)$$

where $P_1$ is the value of the load corresponding to the deformation $x_1$.

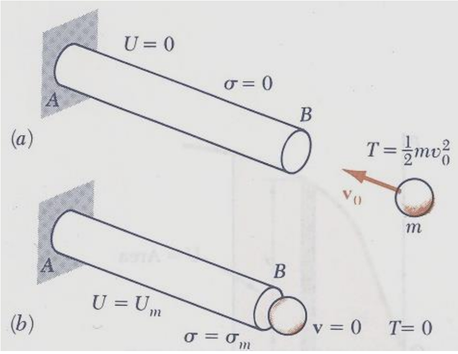


Fig. 3.5

The concept of strain energy is particularly useful in the determination of the effects of **impact loadings** on structures or machine components. Consider, for example, a body of mass m moving with a velocity $V_0$ which strikes the end B of a rod AB (Fig. 2.5a). Neglecting the inertia of the elements of the rod, and assuming no dissipation of energy during the impact, we find that the maximum strain energy $U_m$ acquired by the rod (Fig. 2.5b) is equal to the original kinetic energy of the moving body.

$$T = \frac{1}{2} m V_0^2$$

We may then determine the value $P_m$ of the static load which would have produced the same strain energy in the rod, and obtain the value $\sigma_m$ of the largest stress occurring in the rod by dividing $P_m$ by the cross-sectional area of the rod.

## 3.2. <u>STRAIN-ENERGY DENSITY</u>

We know that the load-deformation diagram for a rod BC depends upon the length L and the cross-sectional area A of the rod. The strain energy U defined by Eq. (2), therefore, will also depend upon the dimensions of the rod. In order to eliminate the effect of size from our discussion and direct our attention to the properties of the material, we shall consider the strain energy per unit volume. Dividing the strain energy U by the volume V = AL of the rod (Fig. 2.1), and using Eq. (2), we have

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

Recalling that $P/A$ represents the normal stress $\sigma_x$ in the rod, and $x/L$ the normal stain $\varepsilon$ , we write

$$\frac{U}{V} = \int_0^{\varepsilon_1} \sigma_x d\varepsilon_x$$

where $\varepsilon_1$ denotes the value of the strain corresponding to the elongation $x_1$. The strain energy per unit volume, $U/V$ , is referred to as the strain-energy density and will be denoted by the letter u. We have, therefore,

**Strain-energy density = u =** $\int_0^{\varepsilon_1} \sigma_x d\varepsilon_x$ …(4)

The strain-energy density u is expressed in units obtained by dividing units of energy by units of volume. Thus the strain-energy density is expressed in $J/m^3$ or its multiples $kJ/m^3$ and $MJ/m^3$ .

Referring to (Fig. 2.6), we note that the strain-energy density u is equal to the area under the stress-strain curve, measured from $\varepsilon_x = 0$ to $\varepsilon_x = \varepsilon_1$. If the material is unloaded, the stress returns to zero, but there is a permanent deformation represented by the strain $\varepsilon_p$ , and only the portion of the strain energy per unit volume corresponding to the triangular area may be recovered. The remainder of the energy spent in deforming the material is dissipated in the form of heat. The value of the strain-energy density obtained by setting $\varepsilon_1 = \varepsilon_R$ in Eq. (4), where $\varepsilon_R$ is the strain at rupture, is known as the **modulus of toughness** of the material. It is equal to the area under the entire stress-strain diagram (Fig. 2.7) and **represents the energy per unit volume required tocause the material to rupture**. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength, and that the capacity of a structure to withstand an impact load depends upon the toughness of the material used.

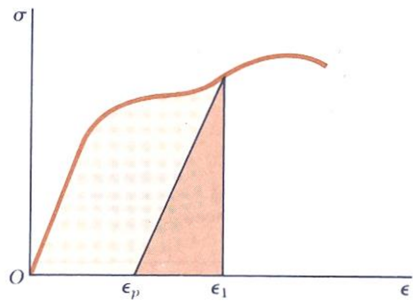

Fig. 3.6

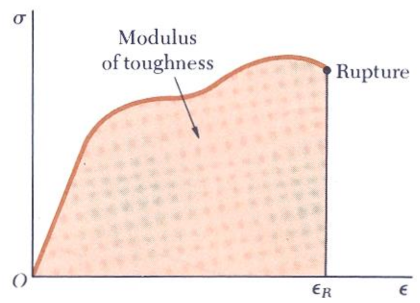

Fig. 3.7

If the stress $\sigma_x$ remains within the proportional limit of the material, Hooke's law applies and we may write

$\sigma_x = E\varepsilon_x$ …(5)

Substituting for $\sigma_x$ from (5) into (4), we have u = $\int_0^{\varepsilon_1} E\varepsilon_x d\varepsilon_x = \frac{E\varepsilon_1^2}{2}$ …(6)

or, using Eq. (5) to express $\varepsilon_1$ in terms of the corresponding stress $\sigma_1$ ,

$$u = \frac{\sigma_1^2}{2E} \qquad \ldots(7)$$

The value $u_y$ of the strain-energy density obtained by setting $\sigma_1 = \sigma_y$ in Eq. (7), where $\sigma_y$ is the yield strength, is called the **modulus of resilience** of the material. We have

$$u_y = \frac{\sigma_y^2}{2E} \ldots(8)$$

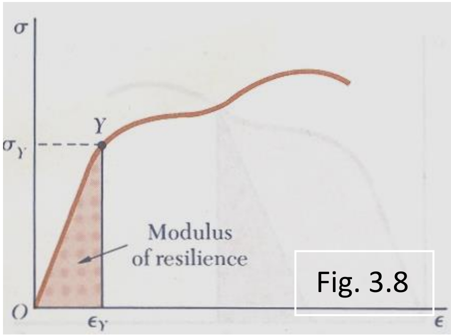

Fig. 3.8

The modulus of resilience is equal to the area under the straight-line portion OY of the stress-strain diagram (Fig. 2.8) and **represents the energy per unit volume that the material may absorb without yielding**. The capacity of a structure to withstand an impact load without being permanently deformed clearly depends upon the resilience of the material used.

Since the modulus of toughness and the modulus of resilience represent characteristic values of the strain-energy density of the material considered, they are both expressed in J/m$^3$ or its multiples.

## 3.3. ELASTIC STRAIN ENERGY FOR NORMAL STRESSES

Since the rod considered in the preceding section was subjected to uniformly distributed stresses $\sigma_x$, the strain-energy density was constant throughout the rod and could be defined as the ratio $U/V$ of the strain energy U and the volume V of the rod. In a structural element or machine part with a non-uniform stress distribution, the strain-energy density u may be defined by considering the strain energy of a small element of material of volume $\Delta$V and writing

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V}$$

or

$$u = \frac{dU}{dV} \ldots(9)$$

The expression obtained for it in Sec. 2.2 in terms of $\sigma_x$ and $\varepsilon_x$ remains valid, i.e., we still have

$$u = \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \qquad \ldots(10)$$

but the stress $\sigma_x$, the strain $\varepsilon_x$, and the strain-energy density u will generally vary from point to point.

For values of $\sigma_x$ within the proportional limit, we may set $\sigma_x = E\varepsilon_x$ in Eq. (10) and write

$$u = \frac{1}{2}E\varepsilon_x^2 = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} \ldots(11)$$

The value of the strain energy U of a body subjected to uniaxial normal stresses may be obtained by substituting for it from Eq. (11) into Eq. (9) and integrating both members. We have

$$U = \int \frac{\sigma_x^2}{2E} dV \ldots(12)$$

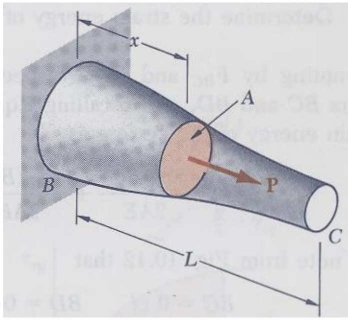

Fig. 3.9

The expression obtained is valid only for elastic deformations and is referred to as the **elastic strain energy** of the body.

***Strain Energy under Axial Loading***: We know that when a rod is subjected to a centric axial loading, the normal stresses $\sigma_x$ may be assumed uniformly distributed in any given transverse section. Denoting by A the area of the section located at a distance x from the end B of the rod (Fig. 2.9), and by P the internal force in that section, we write $\sigma_x = P/A$. Substituting for $\sigma_x$ into Eq. (12), we have

$$U = \int \frac{P^2}{2EA^2} dV$$

or setting dV = A dx,

$$U = \int_0^L \frac{P^2}{2AE} dx \qquad \ldots(13)$$

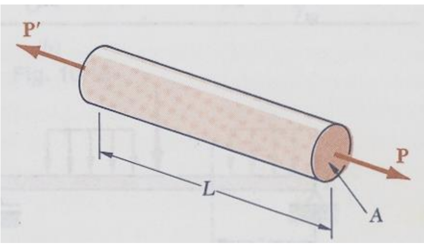

Fig. 3.10

In the case of a rod of uniform cross section subjected at its ends to equal and opposite forces of magnitude P (Fig. 2.10), Eq. (13) yields

$$U = \frac{P^2L}{2AE} \qquad \ldots(14)$$

<u>**EXAMPLES**</u>

**1)** A rod consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 2.11).

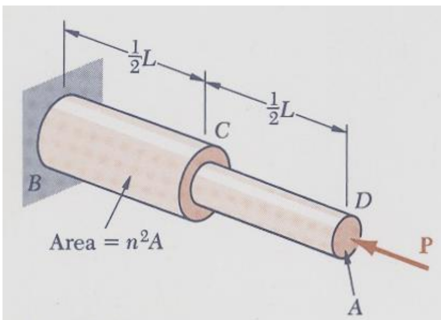

Fig. 3.11

Determine the strain energy of the rod when it is subjected to a centric axial load P, expressing the result in terms of P, L, E, the cross-sectional area A of portion CD, and the ratio n of the two diameters.

**Solution**

We use Eq. (14) to compute the strain energy of each of the two portions, and add the expressions obtained:

$$U_n = \frac{P^2(\frac{1}{2}L)}{2AE} + \frac{P^2(\frac{1}{2}L)}{2(n^2A)E} = \frac{P^2L}{4AE}\left(1 + \frac{1}{n^2}\right)$$

or

$$U_n = \frac{1 + n^2}{2n^2}\frac{P^2L}{2AE} \qquad \text{...(15)}$$

We check that, for n = 1, we have

$$U_1 = \frac{P^2L}{2AE}$$

which is the expression given in Eq. (14) for a rod of length L and uniform cross section of area A. We also note that, for n > 1, we have $U_n < U$ for example, when n = 2, we have $U_2 = \left(\frac{5}{8}\right)U_1$. Since the maximum stress occurs in portion CD of the rod and is equal to $\sigma_{max.} = P/A$, it follows that, for a given allowable stress, increasing the diameter of portion BC of the rod results in a decrease of the overall energy-absorbing capacity of the rod. Unnecessary changes in cross-sectional area should therefore be avoided in the design of members which may be subjected to loadings, such as impact loadings, where the energy-absorbing capacity of the member is critical.

**2)** A load P is supported at B by two rods of the same material and of the same uniform cross section of area A (Fig. 3.12). Determine the strain energy of the system.

**Solution**

Denoting by $F_{BC}$ and $F_{BD}$, respectively, the forces in members BC and BD, and recalling Eq. (14), we express the strain energy of the system as

$$U = \frac{F_{BC}^2 (BC)}{2AE} + \frac{F_{BD}^2 (BD)}{2AE} \quad \dots(16)$$

But we note from Fig. 3.12 that

$$BC = 0.6l \qquad BD = 0.8l$$

and from the free-body diagram of pin B and the corresponding force triangle (Fig. 3.13) that

$$F_{BC} = +0.6P \qquad F_{BD} = -0.8P$$

Substituting into Eq. (16), we have

$$U = \frac{P^2 l[(0.6)^3 + (0.8)^3]}{2AE} = 0.364\frac{P^2 l}{AE}$$

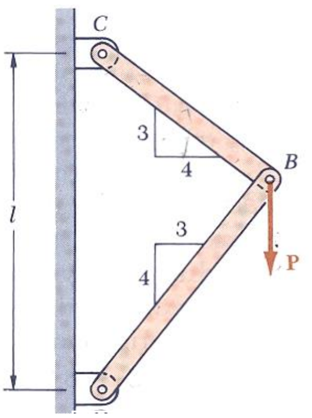


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Fig. 3.12

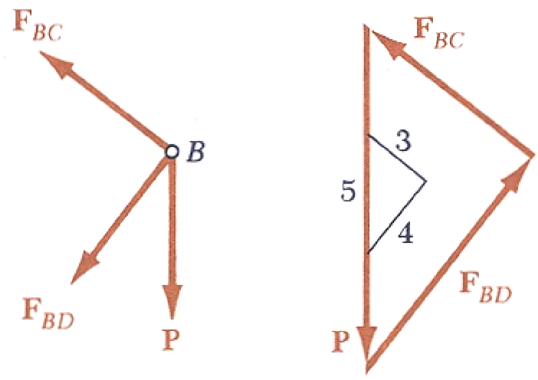



Fig. 3.13

***Strain Energy in Bending***: Consider a beam AB subjected to a given loading (Fig. 3.14), and let M be the bending moment at a distance x from end A . Neglecting for the time being the effect of shear, and taking into account only the normal stresses $\sigma_x = My/I$ , we substitute this expression into Eq. (12) and write

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

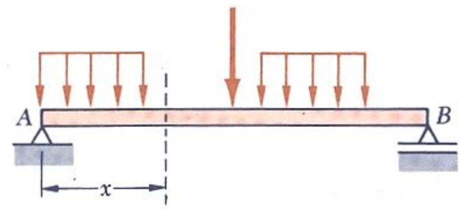



Fig. 3.14

Setting dv = dA dx, where dA represents an element of the cross- sectional area, and recalling that $M^2/2EI^2$ is a function of x alone, we have

$$U = \int_0^L \frac{M^2}{2EI^2}(\int y^2\, dA)\, dx$$

Recalling that the integral within the parentheses represents the moment of inertia I of the cross section about its neutral axis, we write

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad ...(17)$$

**3)** Determine the strain energy of the prismatic cantilever beam AB (Fig. 3.15), taking into account only the effect of the normal stresses.

**<u>Solution</u>**

The bending moment at a distance x from end A is M = -Px . Substituting this expression into Eq. (17), we write

$$U = \int_0^L \frac{P^2x^2}{2EI} dx = \frac{P^2L^3}{6EI}$$

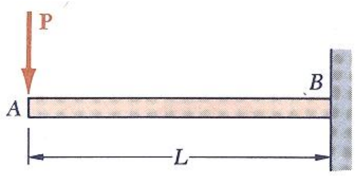


Fig.3.15

### 3.4. <u>ELASTIC STRAIN ENERGY FOR SHEARING STRESSES</u>

When a material is subjected to plane shearing stresses the strain-energy density at a given point may be expressed as

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy} \qquad ...(18)$$

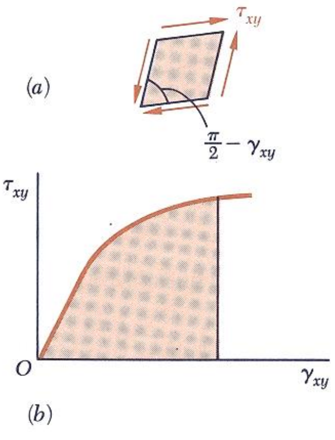


Fig. 3.16

where $\gamma_{xy}$ is the shearing strain corresponding to $\tau_{xy}$ (Fig. 3.16a). We note that the strain-energy density u is equal to the area under the shearing stress-strain diagram (Fig. 2. 16b).

For values of $\tau_{xy}$ within the proportional limit, we have $\tau_{xy} = G\gamma_{xy}$, where G is the modulus of rigidity of the material. Substituting for $\tau_{xy}$ into Eq. (18) and performing the integration, we write

$$u = \frac{1}{2}G\gamma_{xy}^2 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G} \qquad \text{...(19)}$$

The value of the strain energy U of a body subjected to plane shearing stresses may be obtained by recalling from Sec. 3.4 that

$$u = \frac{dU}{dV} \qquad \text{...(9)}$$

Substituting for it from Eq. (19) into Eq. (9) and integrating both members, we have

$$U = \int \frac{\tau_{xy}^2}{2G}dV \qquad \text{...(20)}$$

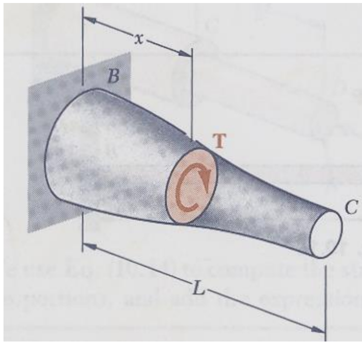


Fig. 3.17

This expression defines the elastic strain associated with the shear deformations of the body. As the similar expression obtained in Sec. 3.4 for uniaxial normal stresses, it is valid only for elastic deformations.

***Strain Energy in Torsion:*** Consider a shaft BC of length L subjected to one or several twisting couples. Denoting by J the polar moment of inertia of the cross section located at a distance x from B (Fig. 3.17), and by T the internal torque in that section, we recall that the shearing stresses in the section are $\tau_{xy} = T\rho/J$. Substituting for $\tau_{xy}$ into Eq. (20), we have

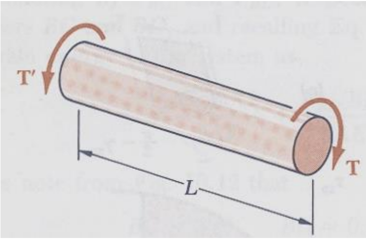
$$U = \int \frac{\tau_{xy}^2}{2G}dV = \int \frac{T^2\rho^2}{2GJ^2}dV$$

Setting dv = dA dx, where dA represents an element of the cross- sectional area, and observing that $T^2/2GJ^2$ is a function of x alone, we write

$$U = \int_0^L \frac{T^2}{2GJ^2}\left(\int \rho^2\, dA\right) dx$$

Recalling that the integral within the parentheses represents the polar moment of inertia J of the cross section, we have

$$U = \int_0^L \frac{T^2}{2GJ} dx \qquad \text{... (21)}$$




In the case of a shaft of uniform cross section subjected at its ends to equal and opposite couples of magnitude T (Fig. 3.18), Eq. (21) yields

$$U = \frac{T^2 L}{2GJ} \qquad \text{... (22)}$$

**4)** A circular shaft consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 3.19). Determine the strain energy of the shaft when it is subjected to a twisting couple T at end D, expressing the result in terms of T, L, G, the polar moment of inertia J of the smaller cross section, and the ratio a of the two diameters.

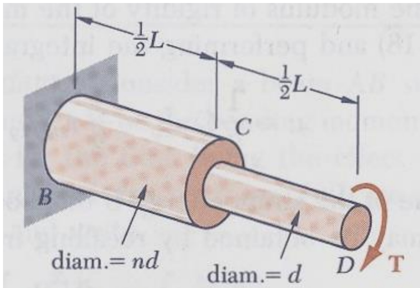



Fig. 3.19

**<u>Solution</u>**

We use Eq. (22) to compute the strain energy of each of the two portions of shaft, and add the expressions obtained. Noting that the polar moment of inertia of portion BC is equal to $n^4J$, we write

$$U_n = \frac{T^2(\frac{1}{2}L)}{2GJ} + \frac{T^2(\frac{1}{2}L)}{2G(n^4J)} = \frac{T^2L}{4GJ}\left(1 + \frac{1}{n^4}\right)$$

or

$$U_n = \frac{1+n^4}{2n^4}\frac{T^2L}{2GJ} \qquad \text{...(23)}$$

We check that, for n = 1, we have

$$U_1 = \frac{T^2L}{2GJ}$$

which is the expression given in Eq. (22) for a shaft of length L and uniform cross section. We also note that, for n > 1, we have $U_n < U_1$ ; for example, when n = 2, we have $U_2 = \left(\frac{17}{32}\right)U_1$.

Since the maximum shearing stress occurs in the portion CD of the shaft and is proportional to the torque T, we note as we did earlier in the case of the axial loading of a rod that, for a givenallowable stress, increasing the diameter of portion BC of the shaft results in a decrease of the overall energy-absorbing capacity of the shaft.

***Strain Energy under Transverse Loading***: In Sec. 3.4 we obtained an expression for the strain energy of a beam subjected to a transverse loading. However, in deriving that expression we took into account only the effect of the normal stresses due to bending and neglected theeffect of the shearing stresses. We shall now take into account the effect of both types of stresses.

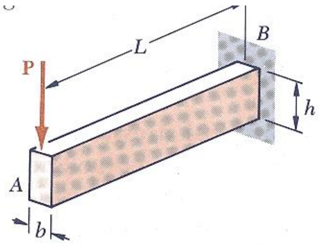


Fig. 3.20

**5)** Determine the strain energy of the rectangular cantilever beam AB (Fig. 3.20), taking into account the effect of both normal and shear stresses.

**Solution**

We first recall from Example 3 that the strain energy due to the normal stresses $\sigma_x$ is

$$U_\sigma = \frac{P^2L^3}{6EI}$$

To determine the strain energy $U_\tau$ due to the shearing stresses we apply the equation below for a beam with a rectangular cross section of width b and depth h,

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right) = \frac{3}{2}\frac{P}{bh}\left(1 - \frac{y^2}{c^2}\right)$$

Substituting for $\tau_{xy}$ into Eq. (20), we write

$$U_\tau = \frac{1}{2G}\left(\frac{3}{2}\frac{P}{bh}\right)^2 \int \left(1 - \frac{y^2}{c^2}\right)^2 dV$$

or, setting dV = b dy dx, and after reductions,

$$U_\tau = \frac{9P^2}{8Gbh^2}\int_{-c}^{c}\left(1-2\frac{y^2}{c^2}+\frac{y^4}{c^4}\right)dy\int_0^L dx$$

Performing the integrations, and recalling that c = h/2, we have

$$U_\tau = \frac{9P^2L}{8Gbh^2}\left[y-\frac{2}{3}\frac{y^3}{c^2}+\frac{1}{5}\frac{y^5}{c^4}\right]_{-c}^{+c} = \frac{3P^2L}{5Gbh} = \frac{3P^2L}{5GA}$$

The total strain energy of the beam is thus

$$U = U_\sigma + U_\tau = \frac{P^2L^3}{6EI}+\frac{3P^2L}{5GA}$$

or, noting that $I/A = h^2/12$ and factoring the expression for $U_\sigma$,

$$U = \frac{P^2L^3}{6EI}\left(1+\frac{3Eh^2}{10GL^2}\right) = U_\sigma\left(1+\frac{3Eh^2}{10GL^2}\right) \qquad \text{...(24)}$$

We know that $G \geq E/3$, we conclude that the parenthesis in the expression obtained is less than $1+0.9(h/L)^2$ and, thus, that the relative error is less than $0.9(h/L)^2$ when the effect of shear is neglected. For a beam with a ratio $h/L$ less than $1/10$, the percentage error is less than 0.9%. It is therefore customary in engineering practice to neglect the effect of shear in computing the strain energy of slender beams.

## 3.5. STRAIN ENERGY FOR A GENERAL STATE OF STRESS

In the preceding sections, we determined the strain energy of a body in a state of uniaxial stress (Sec. 2.3) and in a state of plane shearing stress (Sec. 2.4). In the case of a body in a general state of stress characterized by the six stress components $\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{xy}, \tau_{yz},\ and\ \tau_{zx}$ , the strain-energy density may be obtained by adding the expressions given in Eqs. (10) and (18), as well as the four other expressions obtained through a permutation of the subscripts. The strain energy density of a three-dimensional principal stress system is given by:

$$u = \frac{1}{2E}[\sigma_a^2+\sigma_b^2+\sigma_c^2-2\nu(\sigma_a\sigma_b+\sigma_b\sigma_c+\sigma_c\sigma_a)] \qquad \text{...(25)}$$

where $\sigma_a$, $\sigma_b$, and $\sigma_c$ are the principal stresses at the given point.

This total strain energy can be conveniently considered as made up of two parts:

(a) the volumetric or dilatational strain energy density;

(b) the shear or distortional strain energy density.

Therefore we write

$$u = u_v + u_d$$

**(a) <u>The volumetric or dilatational strain energy density</u>**

This is the strain energy density associated with a mean or hydrostatic stress of $\frac{1}{3}(\sigma_a + \sigma_b + \sigma_c) = \bar{\sigma}$ acting equally in all three mutually perpendicular directions giving rise to no distortion, merely a change in volume. And we can set also

$$\sigma_a = \bar{\sigma} + \sigma'_a \qquad \sigma_b = \bar{\sigma} + \sigma'_b \qquad \sigma_c = \bar{\sigma} + \sigma'_c$$

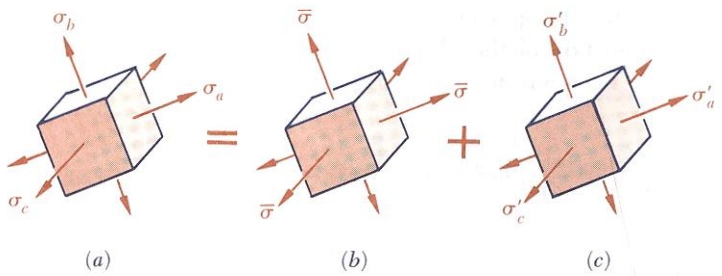



Fig. 3.21

$\therefore$ Volumetric strain energy density ($u_v$) = $\frac{(1-2\nu)}{6E}\left[(\sigma_a + \sigma_b + \sigma_c)^2\right]$ …(26)

**(b) <u>The shear or distortional strain energy density</u>**

In order to consider the general principal stress case it is necessary, to add to the mean stress $\bar{\sigma}$ in the three perpendicular directions, certain so-called deviatoric stress values to return the stress system to values of $\sigma_a$, $\sigma_b$ and $\sigma_c$. These deviatoric stresses are then associated directly with change of shape, i.e. distortion, without change in volume and the strain energy associated with this mechanism is given by

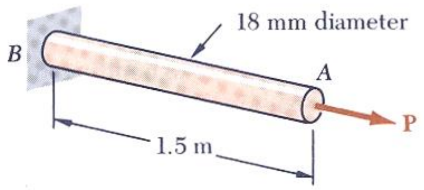
$$u_d = \frac{1}{12G}[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2]$$

In the case of plane stress, and assuming that the c axis is perpendicular to the plane of stress, we have $\sigma_c = 0$ and Eq. (27) reduces to

$$u_d = \frac{1}{6G}(\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2) \qquad \text{...(27)}$$

**EXAMPLES**

**6)** During a routine manufacturing operation, rod AB must acquire an elastic strain energy of 12J. Using E = 200 GPa, determine the required yield strength of the steel if the factor of safety with respect to permanent deformation is to be five.




**Design strain energy** : Since a factor of safety of five is required, the rod should be designed for a strain energy of

$$U = 5(12\text{ J}) = 60\text{ J}$$

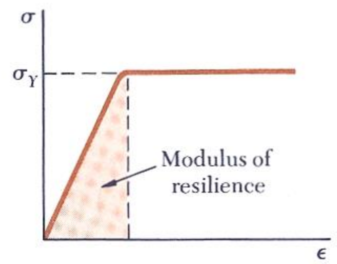
**Strain-Energy Density**: The volume of the rod is

$$V = AL = \frac{\pi}{4}(0.018\text{ m})^2(1.5\text{ m}) = 381.7 \times 10^{-6}\text{ m}^3$$

Since the rod is of uniform cross section, the required strain-energy density is

$$u = \frac{U}{V} = \frac{60\text{ J}}{381.7 \times 10^{-6}\text{ m}^3} = 157.2 \times 10^3\text{ J/m}^3$$

**Yield Strength:** We recall that the modulus of resilience is equal to the strain-energy density when the maximum stress is equal to $\sigma_y$. Using Eq. (8),we write

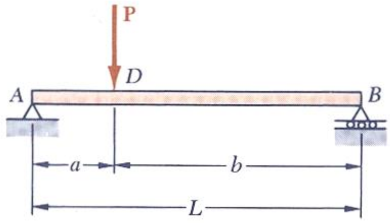



$$u = \frac{\sigma_Y^2}{2E}$$

$$157.2 \times 10^3\text{ J/m}^3 = \frac{\sigma_Y^2}{2(200\text{ GPa})}$$

$\sigma_y$ = 250.8 MPa = 251

**Comment**




It is important to note that, since energy loads are not linearly related to the stresses they produce, factors of safety associated with energy loads should be applied to the energy loads and not to the stresses.

**7)** (**a**) Taking into account only the effect of normal stresses due to bending, determine the strain energy of the prismatic beam AB for the loading shown. (**b**) Evaluate the strain energy, knowing that the beam is a W 250 x 67, P = 180 kN, L = 3.6 m, a = 0.9 m, b = 2.7 m and E = 200 GPa.

**Bending Moment :** Using the free-body diagram of the entire beam, we determine the reactions

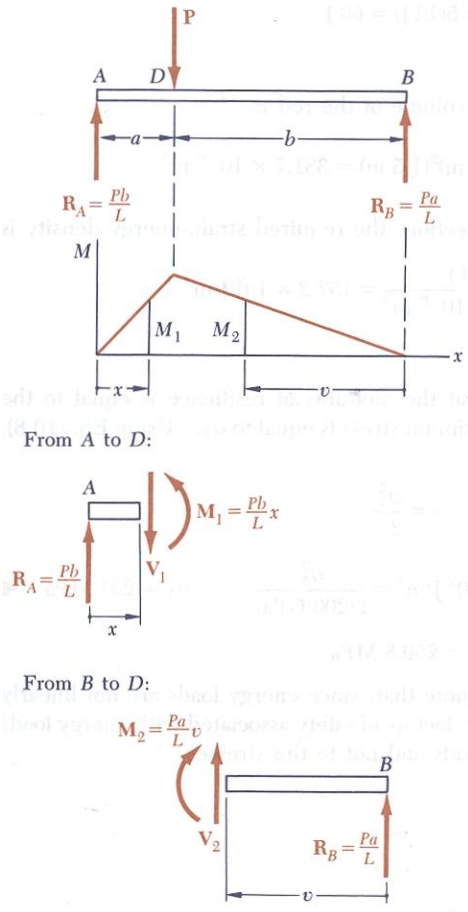
$$\mathbf{R}_A = \frac{Pb}{L}\uparrow \qquad \mathbf{R}_B = \frac{Pa}{L}\uparrow$$

For portion AD of the beam, the bending moment is

$$M_1 = \frac{Pb}{L}x$$

For portion DB, we note that the bending moment at a distance v from end B is

$$M_2 = \frac{Pa}{L}v$$




**(a) Strain Energy:** Since strain energy is a scalar quantity, we add the strain energy of portion AD to that of portion DB to obtain the total strain energy of the beam. Using Eq. (17), we write

$$U = U_{AD} + U_{DB}$$
$$= \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}v\right)^2 dv$$
$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

or, since (a + b) = L, $$U = \frac{P^2a^2b^2}{6EIL}$$

**(b) Evaluation of the Strain Energy:** The moment of inertia of a W 250 x 67 rolled-steel shape is obtained from Appendix C and the given data is repeated here:

$$P = 180 \text{ kN} \qquad L = 3.6 \text{ m}$$
$$a = 0.9 \text{ m} \qquad b = 2.7 \text{ m}$$
$$E = 200 \text{ GPa} \qquad I = 103.2 \times 10^{-6} \text{ m}^4$$

Substituting into the expression for U, we have

$$U = \frac{(180 \text{ kN})^2(0.9 \text{ m})^2(2.7 \text{ m})^2}{6(200 \text{ GPa})(103.2 \times 10^{-6} \text{ m}^4)(3.6 \text{ m})}$$

U = 429 J

## 3.6. IMPACT LOADING

Consider a rod BD of uniform cross section which is hit at its end B by a body of mass **m** moving with a velocity $v_o$ (Fig. 3.22a). As the rod deforms under the impact (Fig. 3.22b), stresses develop within the rod and reach a maximum value $\sigma_m$. After vibrating for a while, the rod will come to rest, and all stresses will disappear. Such a sequence of events is referred to as an **impact loading**.

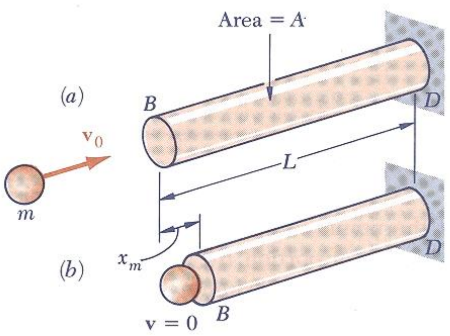


Fig. 3.22

In order to determine the maximum value $\sigma_m$ of the stress occurring at a given point of a structure subjected to an impact loading, we shall make several simplifying assumptions.

First, we shall assume that the kinetic energy T = $\frac{1}{2}mV_o^2$ of the striking body is transferred entirely to the structure and, thus, that the strain energy $U_m$ corresponding to the maximum deformation $x_m$ is

$$U_m = \tfrac{1}{2}mv_0^2 \qquad \text{...(28)}$$

This assumption leads to the following two specific requirements:

1. No energy should be dissipated during the impact.
2. The striking body should not bounce off the structure and retain part of its energy. This, in turn, necessitates that the inertia of the structure be negligible, compared to the inertia of the striking body.

In practice, neither of these requirements is satisfied, and only part of the kinetic energy of the striking body is actually transferred to the structure. Thus, assuming that all of the kinetic energy of the striking body is transferred to the structure leads to a conservative design of that structure.

We shall further assume that the stress-strain diagram obtained from a static test of the material is also valid under impact loading. Thus, for an elastic deformation of the structure, we may express the maximum value of the strain energy as

$$U_m = \int \frac{\sigma_m^2}{2E} dV \qquad \text{...(29)}$$

In the case of the uniform rod of Fig. 3.22, the maximum stress $\sigma_m$ has the same value throughout the rod, and we may write $U_m = \sigma_m^2 V / 2E$. Solving for $\sigma_m$ and substituting for $U_m$ from Eq.(28), we write

$$\sigma_m = \sqrt{\frac{2U_m E}{V}} = \sqrt{\frac{m v_0^2 E}{V}} \qquad \text{...(30)}$$

We note from the expression obtained that selecting a rod with a large volume V and a low modulus of elasticity E will result in a smaller value of the maximum stress $\sigma_m$ for a given impact loading.

In most problems, the distribution of stresses in the structure is not uniform, and formula (30) does not apply. It is then convenient to determine the static load $P_m$ which would produce the same strain energy as the impact loading, and compute from $P_m$ the corresponding value $\sigma_m$ of the largest stress occurring in the structure.

**EXAMPLES**

**8)** A body of mass **m** moving with a velocity $\mathbf{v_0}$ hits the end B of the non-uniform rod BCD (Fig. 3.23). Knowing that the diameter of portion BC is twice the diameter of portion CD, determine the maximum value $\boldsymbol{\sigma_m}$ of the stress in the rod.

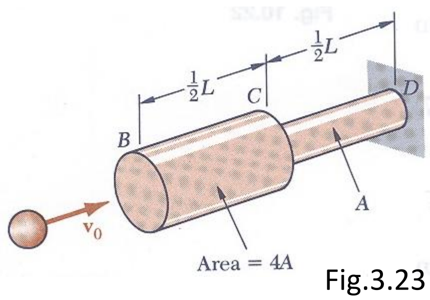



Fig.3.23

**Solution**

Making n = 2 in the expression (15) obtained in Example 1, we find that when rod BCD is subjected to a static load $P_m$ its strain energy is

$$U_m = \frac{5P_m^2 L}{16AE} \qquad \text{...(31)}$$

where A is the cross-sectional area of portion CD of the rod.

Solving Eq. (31) for $P_m$ we find that the static load which produces in the rod the same strain energy as the given impact loading is

$$P_m = \sqrt{\frac{16}{5}\frac{U_m AE}{L}}$$

where $U_m$ is given by Eq. (28). The largest stress occurs in portion CD of the rod. Dividing $P_m$ by the area A of that portion, we have

$$\sigma_m = \frac{P_m}{A} = \sqrt{\frac{16}{5}\frac{U_m E}{AL}} \qquad \text{...(32)}$$

or, substituting for $U_m$ from Eq. (28),

$$\sigma_m = \sqrt{\frac{8}{5}\frac{mv_0^2 E}{AL}} = 1.265\sqrt{\frac{mv_0^2 E}{AL}}$$

Comparing this value with the value obtained for $U_m$ in the case of the uniform rod of Fig. 3.22 and making V = AL in Eq. (30), we note that the maximum stress in the rod of variable cross section is 26.5% larger than in the lighter uniform rod. Thus, as we observed earlier in our discussion of Example 1, increasing the diameter of portion BC of the rod results in a decrease of the energy-absorbing capacity of the rod.

**9)** A block of weight W is dropped from a height h onto the free end of the cantilever beam AB (Fig. 3.24). Determine the maximum value of the stress in the beam.

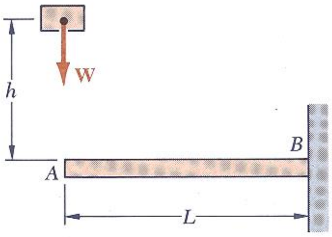



Fig.3.24

**Solution**

As it falls through the distance ***h***, the potential energy ***Wh*** of the block is transformed into kinetic energy. As a result of the impact, the kinetic energy in turn is transformed into strain energy. We have, therefore,

$$U_m = Wh \qquad \text{...(33)}$$

Recalling the expression obtained for the strain energy of the cantilever beam AB in Example 3 and neglecting the effect of shear, we write

$$U_m = \frac{P_m^2 L^3}{6EI}$$

Solving this equation for $P_m$, we find that the static force which produces in the beam the same strain energy is

$$P_m = \sqrt{\frac{6U_m EI}{L^3}} \qquad \text{...(34)}$$

The maximum stress $\sigma_m$ occurs at the fixed end B and is equal to

$$\sigma_m = \frac{|M|c}{I} = \frac{P_m Lc}{I}$$

Substituting for $P_m$ from (34), we write

$$\sigma_m = \sqrt{\frac{6U_m E}{L(I/c^2)}} \qquad \text{...(35)}$$

or, recalling (33),

$$\sigma_m = \sqrt{\frac{6WhE}{L(I/c^2)}}$$

## 3.7. DESIGN FOR IMPACT LOADS

We shall now compare the values obtained in the preceding section for the maximum stress $\sigma_m$ (a) in the rod of uniform cross section of Fig. 3.22, (b) in the rod of variable cross section of Example 8, and (c) in the cantilever beam of Example 9, assuming that the latter has a circular cross section of radius c.

(a) We first recall from Eq. (30) that, if $U_m$ denotes the amount of energy transferred to the rod as a result of the impact loading, the maximum stress in the rod of uniform cross section is

$$\sigma_m = \sqrt{\frac{2U_m E}{V}} \qquad \text{...(36a)}$$

where V is the volume of the rod.

(b) Considering next the rod of Example 8 and observing that the volume of the rod is

$$V = 4A(L/2) + A(L/2) = 5AL/2$$

we substitute AL = 2V/5 into Eq. (32) and write

$$\sigma_m = \sqrt{\frac{8U_m E}{V}} \qquad \text{...(36b)}$$

c) Finally, recalling that $I = \frac{1}{4}\pi c^4$ for a beam of circular cross section, we note that

$$L(I/c^2) = L(\tfrac{1}{4}\pi c^4/c^2) = \tfrac{1}{4}(\pi c^2 L) = \tfrac{1}{4}V$$

where V denotes the volume of the beam. Substituting into Eq. (35), we express the maximum stress in the cantilever beam of Example 9 as

$$\sigma_m = \sqrt{\frac{24U_m E}{V}} \qquad \text{...(36c)}$$

We note that, in each case, the maximum stress $\sigma_m$ is proportional to the square root of the modulus of elasticity of the material and inversely proportional to the square root of the volume of the member. Assuming all three members to have the same volume and to be of the same material, we also note that, for a given value of the absorbed energy, the uniform rod will experience the lowest maximum stress, and the cantilever beam the highest one.

This observation may be explained by the fact that, the distribution of stresses being uniform in case **(a),** the strain energy will be uniformly distributed throughout the rod. In case **(b),** on the other hand, the stresses in portion BC of the rod are only 25% as large as the stresses in portion CD. This uneven distribution of the stresses and of the strain energy results in a maximum stress $\sigma_m$ twice as large as the corresponding stress in the uniform rod. Finally, in case (**c**), where the cantilever beam is subjected to a transverse impact loading, the stresses vary linearly along the beam as well as across a transverse section. The very uneven resulting distribution of strain energy causes the maximum stress $\sigma_m$ to be 3.46 times larger than if the same member had been loaded axially as in case **(a).**

The properties noted in the three specific cases discussed in this section are quite general and may be observed in all types of structures and impact loadings. We thus conclude that a structure designed to withstand effectively an impact load should

1. Have a large volume
2. Be made of a material with a low modulus of elasticity and a high yield strength
3. Be shaped so that the stresses are distributed as evenly as possible throughout the structure

## 3.8. WORK AND ENERGY UNDER A SINGLE LOAD

When we first introduced the concept of strain energy at the beginning of this chapter, we considered the work done by an axial load P applied to the end of a rod of uniform cross section (Fig. 2.1). We defined the strain energy of the rod for an elongation $x_1$ as the work of the load P as it is slowly increased from 0 to the value $P_1$ corresponding to $x_1$. We wrote

$$\text{Strain energy} = U = \int_0^{x_1} P\,dx \qquad \text{...(2)}$$

In the case of an elastic deformation, the work of the load P, and thus the strain energy of the rod, were expressed as

$$U = \tfrac{1}{2}P_1x_1 \qquad \text{...(3)}$$

However, when a structure or member is subjected to a single concentrated load, it is possible to use Eq. (3) to evaluate its elastic strain energy, provided that the relation between the load and the resulting deformation is known. For instance, in the case of the cantilever beam of Example 3 (Fig. 3.25), we write

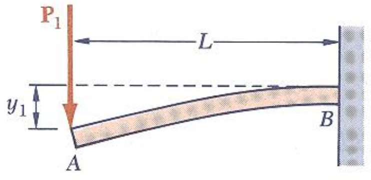


Fig. 3.25

$$U = \tfrac{1}{2}P_1y_1$$

and, substituting for y the value obtained from the table of Beam Deflections and Slopes we have

$$U = \frac{1}{2}P_1\left(\frac{P_1L^3}{3EI}\right) = \frac{P_1^2L^3}{6EI} \qquad ...(37)$$

A similar approach may be used to determine the strain energy of a structure or member subjected to a single couple. Recalling that the elementary work of a couple of moment M is M dθ, where dθ is a small angle. Since M and θ are linearly related the elastic strain energy of a cantilever beam AB subjected to a single couple $M_1$ at its end A (Fig. 3.26) may be expressed as

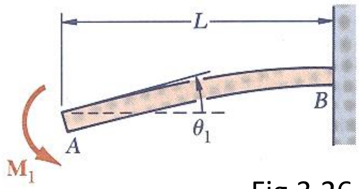


Fig.3.26

$$U = \int_0^{\theta_1} M\, d\theta = \tfrac{1}{2}M_1\theta_1 \qquad ...(38)$$

where $\theta_1$ is the slope of the beam at A. Substituting for $\theta_1$ the value obtained fromthe table of Beam Deflections and Slopes, we write

$$U = \frac{1}{2}M_1\left(\frac{M_1L}{EI}\right) = \frac{M_1^2L}{2EI} \qquad ...(39)$$

In a similar way, the elastic strain energy of a uniform circular shaftAB of length L subjected at its end B to a single torque $T_1$ (Fig. 3.27)may be expressed as

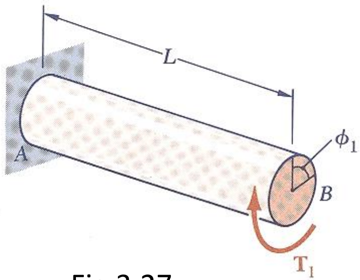


Fig.3.27

$$U = \int_0^{\phi_1} T\, d\phi = \tfrac{1}{2}T_1\phi_1 \qquad ...(40)$$

Substituting for the angle of twist $\phi_1 = \dfrac{T_1L}{JG}$, we verify that

$$U = \frac{1}{2}T_1\left(\frac{T_1L}{JG}\right) = \frac{T_1^2L}{2JG}$$

**Example**

**10.** A block of mass m moving with a velocity $V_o$ hits squarely the prismatic member AB at its midpoint C (Fig. 3.28). Determine (a) the equivalent static load $P_m$, (b) the maximum stress $\sigma_m$ in the member, and (c) the maximum deflection $x_m$ at point C.

**(a) Equivalent Static Load**: The maximum strain energy of the member is equal to the kinetic energy of the block before impact. We have

$$U_m = \tfrac{1}{2} m v_0^2 \qquad ...(41)$$

On the other hand, expressing **$U_m$ as the work of the equivalent horizontal static load** as it is slowly applied at the midpoint C of the member, we write

$$U_m = \tfrac{1}{2} P_m x_m \qquad ...(42)$$

where $x_m$ is the deflection of C corresponding to the static load $P_m$. From the table of Beam Deflections and Slopes , we find that

$$x_m = \frac{P_m L^3}{48EI} \qquad ...(43)$$

Substituting for $x_m$ from (43) into (42), we write

$$U_m = \frac{1}{2} \frac{P_m^2 L^3}{48EI}$$

Solving for $P_m$ and recalling Eq. (41), we find that the static load equivalent to the given impact loading is

$$P_m = \sqrt{\frac{96 U_m EI}{L^3}} = \sqrt{\frac{48 m v_0^2 EI}{L^3}} \qquad ...(44)$$

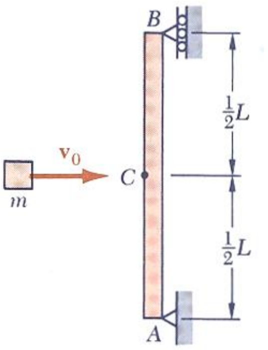



Fig. 3.28

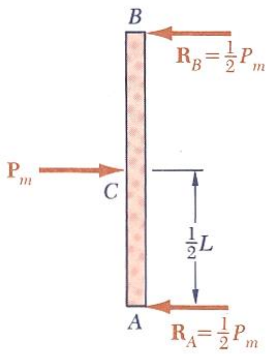



Fig. 3.29

**(b) Maximum Stress:** Drawing the free-body diagram of the member (Fig. 3.29), we find that the maximum value of the bending moment occurs at C and is $M_{max} = P_m L/4$. The maximum stress, therefore, occurs in a transverse section through C and is equal to

$$\sigma_m = \frac{M_{\max} c}{I} = \frac{P_m L c}{4I}$$

Substituting for $P_m$ from (44), we write

$$\sigma_m = \sqrt{\frac{3mv_0^2 EI}{L(I/c)^2}}$$

**(c) Maximum Deflection**: Substituting into Eq. (43) the expression obtained for $P_m$ in (44), we have

$$x_m = \frac{L^3}{48EI}\sqrt{\frac{48mv_0^2 EI}{L^3}} = \sqrt{\frac{mv_0^2 L^3}{48EI}}$$

## 3.9. DEFLECTION UNDER A SINGLE LOAD BY THE WORK-ENERGY METHOD

We saw in the preceding section that, if the deflection $x_1$ of a structure or member under a single concentrated load $P_1$ is known, the corresponding strain energy U may be obtained by writing

$$U = \tfrac{1}{2} P_1 x_1 \qquad \text{...(3)}$$

A similar expression may be used to obtain the strain energy of a structural member under a single couple $M_1$

$$U = \tfrac{1}{2} M_1 \theta_1 \qquad \text{...(38)}$$

Conversely, if the strain energy U of a structure or member subjected to a single concentrated load $P_1$ or couple $M_1$ is known, Eq. (3) or (38) may be used to determine the corresponding deflection $x_1$ or angle $\theta_1$. In order to determine the deflection under a single load applied to a structure consisting of several component parts, we may find it easier, to first compute the strain energy of the structure by integrating the strain-energy density over its various parts, as was done in Secs. 3.4 and 3.5, and then use either

Eq. (3) or Eq. (38) to obtain the desired deflection. Similarly, the angle of twist of a composite shaft may be obtained by integrating the strain-energy density over the various parts of the shaft and solving Eq. (40) for $\phi_1$.

It should be kept in mind that the method presented in this section may be used only if the given structure is subjected to a single concentrated load or couple.

## Example

**11**. A load P is supported at B by two uniform rods of the same cross-sectional area A (Fig. 3.30). Determine the verticaldeflection of point B.

**Solution**

The strain energy of the systemunder the given load was determined in Example 2. Equating the expression obtained for U to the work of the load, we write

$$U = 0.364\frac{P^2 l}{AE} = \frac{1}{2}Py_B$$

and, solving for the vertical deflection of B,

$$y_B = 0.728\frac{Pl}{AE}$$

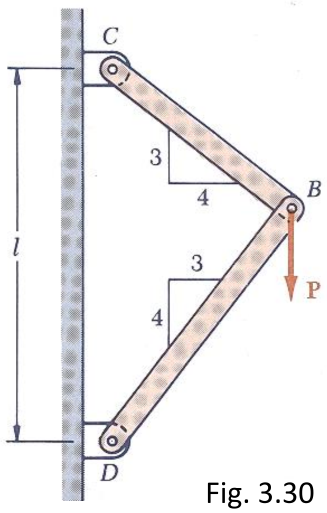


Fig. 3.30

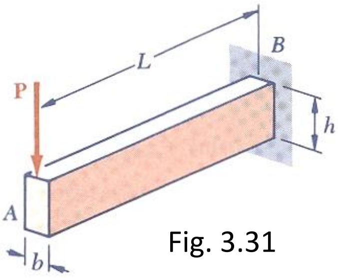


Fig. 3.31

**12.**Determine the deflection of end A of the cantilever beam AB (Fig. 3.31), taking into account the effect of(a) the normal stresses only, (b) both the normal and shearing stresses.

**Solution**

**(a) Effect of Normal Stresses:** The work of the force P as it is slowly applied to A is

$$U = \tfrac{1}{2}Py_A$$

Substituting for U the expression obtained for the strain energy of the beam in Example 3, where only the effect of thenormal stresses was considered, we write

$$\frac{P^2L^3}{6EI} = \frac{1}{2}Py_A$$

and, solving for $y_A$,

$$y_A = \frac{PL^3}{3EI}$$

**(b) Effect of Normal and Shearing Stresses :** We now substitute for U the expression (24) obtained in Example 5, where the effects of both the normal and shearingstresses were taken into account. We have

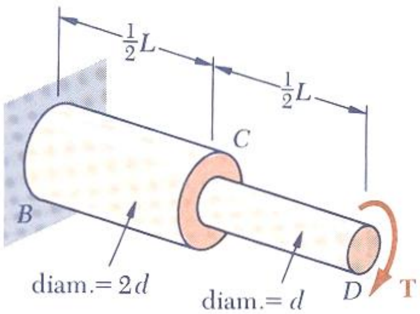
$$U = \frac{P^2L^3}{6EI}\left(1+\frac{3Eh^2}{10GL^2}\right) = U_\sigma\left(1+\frac{3Eh^2}{10GL^2}\right) \quad (\ldots 24) \quad ; \quad \frac{P^2L^3}{6EI}\left(1+\frac{3Eh^2}{10GL^2}\right) = \frac{1}{2}Py_A$$

and, solving for $y_A$,

$$y_A = \frac{PL^3}{3EI}\left(1+\frac{3Eh^2}{10GL^2}\right)$$

We note that the relative error when the effect of shear is neglected is the same that was obtained in Example 5, i.e., less than $0.9(h/L)^2$. As we indicated then, this is less than 0.9% for a beam with a ratio h/L less than 1/10.

**12.**A torque T is applied at the end D of shaft BCD (Fig. 2.32). Knowing that both portions of the shaft are of the same material and same length, but that the diameter of BC is twice the diameter of CD, determine the angle of twist for the entire shaft.




**<u>Solution</u>**

The strain energy of a similar shaft was determined in Example 4 by breaking the shaft into its component parts BC and CD. Making n = 2 in Eq. (23), we have

$$U_n = \frac{1+n^4}{2n^4}\frac{T^2L}{2GJ} \quad (\ldots 23) \; ; \quad U = \frac{17}{32}\frac{T^2L}{2GJ}$$

where G is the modulus of rigidity of the material, and J the polar moment of inertia of portion CD of the shaft. Setting U equal to the work of the torque as it is slowly applied and recalling Eq. (40), to end D, we write

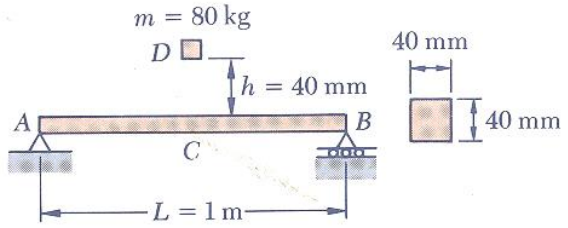
$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2}T_1\phi_1 \quad (\ldots 40) \quad ; \quad \frac{17}{32}\frac{T^2L}{2GJ} = \frac{1}{2}T\phi_{D/B}$$

and, solving for the angle of twist $\phi_{D/B}$ ,

v

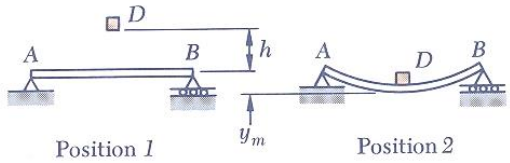
$$\phi_{D/B} = \frac{17TL}{32GJ}$$

**Sample Problems**

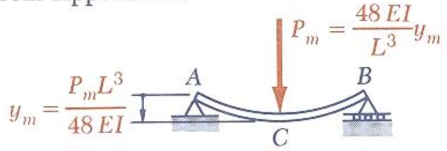



**1.** The block D of mass m is released from rest and falls a distance h before it strikes the midpoint C of the aluminum beam AB. Using E = 70 GPa, determine (a) the maximum deflection of point C, (b) the maximum stress which occurs in the beam.

## Solution




From Appendix D




**Principle of Work and Energy:** Since the block is released from rest, we note that in position1 both the kinetic energy and the strain energy are zero. Inposition 2, where the maximum deflection $y_m$ occurs, the kinetic energy is again zero. Referring to the table of Beam Defections and Slopes we find the expression for $y_m$ shown. The strain energy of the beam in position 2 is

$$U_2 = \frac{1}{2}P_m y_m = \frac{1}{2}\frac{48EI}{L^3}y_m^2 \qquad U_2 = \frac{24EI}{L^3}y_m^2$$

We observe that the work done by the weight W of the block is $W(h + y_m)$. Equating the strain energy of the beam to the work done by W, we have

$$\frac{24EI}{L^3}y_m^2 = W(h + y_m) \qquad \text{...(1)}$$

**(a) Maximum Deflection of Point C**: From the given data we have

$$EI = (70 \times 10^9 \text{ Pa})\tfrac{1}{12}(0.04 \text{ m})^4 = 14.93 \times 10^3 \text{ N} \cdot \text{m}^2$$

$$L = 1 \text{ m} \qquad h = 0.040 \text{ m} \qquad W = mg = (80 \text{ kg})(9.81 \text{ m/s}^2) = 784.8 \text{ N}$$

Substituting into Eq. (1), we have

$$\frac{24(14.93 \times 10^3)}{(1)^3}y_m^2 - 784.8(0.040 + y_m) = 0$$

$$(358.3 \times 10^3)y_m^2 - 784.8y_m - 31.39 = 0$$

Solving this quadratic equation, we find

$$y_m = 10.52 \text{ mm}$$

**(b) Maximum Stress:** The value of $P_m$ is

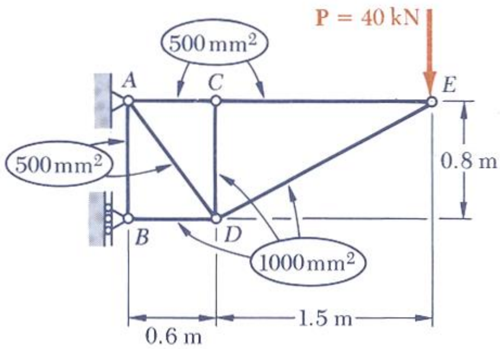
$$P_m = \frac{48EI}{L^3} y_m = \frac{48(14.93 \times 10^3 \text{ N} \cdot \text{m})}{(1 \text{ m})^3}(0.01052 \text{ m}) \qquad P_m = 7540 \text{ N}$$

Since $M_{max} = \frac{1}{4} P_m L$, the maximum stress is

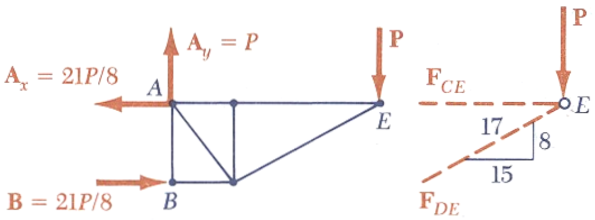
$$\sigma_m = \frac{M_{max}c}{I} = \frac{(\frac{1}{4}P_m L)c}{I} = \frac{\frac{1}{4}(7540 \text{ N})(1 \text{ m})(0.020 \text{ m})}{\frac{1}{12}(0.040 \text{ m})^4}$$

$$\sigma_m = 176.7 \text{ MPa}$$

**2.** Members of the truss shown consist of sections of aluminum pipe with the cross- sectional areas indicated. Using E = 70 GPa, determine the vertical deflection of point E caused by the load P.




**Axial Forces in Truss Members:** The reactions are found by using the free- body diagram of the entire truss. We then consider in sequence the equilibrium of joints, E, C, D, and B. At each joint we determine the forces indicated by dashed lines. At joint B, the equation $\sum F_x = 0$ provides a check of our computations.

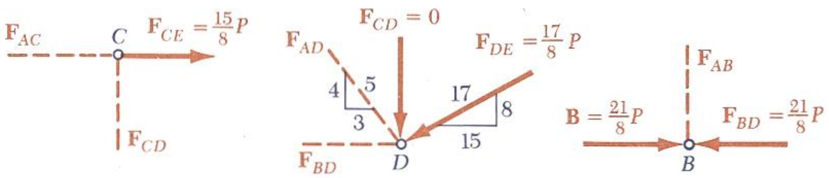



$$\Sigma F_y = 0: F_{DE} = -\frac{17}{8}P \qquad \Sigma F_x = 0: F_{AC} = +\frac{15}{8}P$$

$$\Sigma F_x = 0: F_{CE} = +\frac{15}{8}P \qquad \Sigma F_y = 0: F_{CD} = 0$$

$$\Sigma F_y = 0: F_{AD} = +\frac{5}{4}P$$

$$\Sigma F_x = 0: F_{BD} = -\frac{21}{8}P$$




$$\Sigma F_y = 0: F_{AB} = 0$$

$$\Sigma F_x = 0: \text{(Checks)}$$

**Strain Energy:** Noting that E is the same for all members, we express the strain energy of the truss as follows

$$U = \sum \frac{F_i^2 L_i}{2A_i E} = \frac{1}{2E} \sum \frac{F_i^2 L_i}{A_i} \qquad (1)$$

where Fi is the force in a given member as indicated in the following table and where the summation is extended over all members of the truss.

| Member | $F_i$ | $L_i$, m | $A_i$, m² | $\frac{F_i^2 L_i}{A_i}$ |
|---|---|---|---|---|
| $AB$ | 0 | 0.8 | $500 \times 10^{-6}$ | 0 |
| $AC$ | $+15P/8$ | 0.6 | $500 \times 10^{-6}$ | $4\,219P^2$ |
| $AD$ | $+5P/4$ | 1.0 | $500 \times 10^{-6}$ | $3\,125P^2$ |
| $BD$ | $-21P/8$ | 0.6 | $1000 \times 10^{-6}$ | $4\,134P^2$ |
| $CD$ | 0 | 0.8 | $1000 \times 10^{-6}$ | 0 |
| $CE$ | $+15P/8$ | 1.5 | $500 \times 10^{-6}$ | $10\,547P^2$ |
| $DE$ | $-17P/8$ | 1.7 | $1000 \times 10^{-6}$ | $7\,677P^2$ |

$$\sum \frac{F_i^2 L_i}{A_i} = 29\,700P^2$$

Returning to Eq. (1), we have U = (1/2E)(29.7 X $10^3P^2$).

**Principle of Work-Energy:** We recall that the work done by the load P as it is gradually applied is $\frac{1}{2}Py_E$. Equating the work done by P to the strain energy U and recalling that E = 70 GPa and P = 40 kN, we have

$$\frac{1}{2}Py_E = U \qquad \frac{1}{2}Py_E = \frac{1}{2E}(29.7 \times 10^3 P^2)$$

or

$$y_E = \frac{1}{E}(29.7 \times 10^3 P) = \frac{(29.7 \times 10^3)(40 \times 10^3)}{70 \times 10^9}$$

$$y_E = 16.97 \times 10^{-3}\text{ m} \qquad y_E = 16.97\text{ mm}\downarrow$$

## 3.10. <u>WORK AND ENERGY UNDER SEVERAL LOADS</u>

In this section, we shall see how the strain energy of a structure subjected to several loads may be expressed in terms of the loads and the resulting deflections.

Consider an elastic beam AB subjected to two concentrated loads $F_1$ and $F_2$. The strain energy of the beam is equal to the work of $P_1$ and $P_2$as they are slowly applied to the beam at $C_1$ and $C_2$, respectively (Fig. 3.33). However, in order to evaluate this work, we must first express the deflections $x_1$ and $x_2$ in terms of the loads $P_1$ and $P_2$.

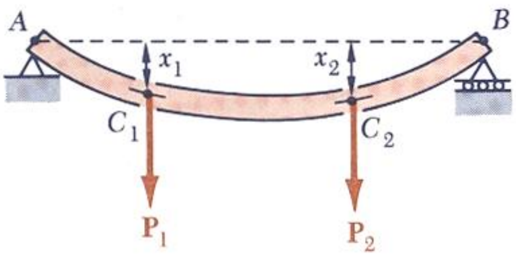



Fig. 3.33

Let us assume that only P1 is applied to the beam (Fig. 3.34). We note that both $C_1$ and $C_2$ are deflected and that

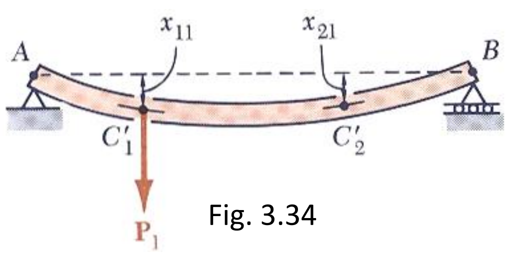



Fig. 3.34

their deflections areproportional to the load $P_1$. Denoting these deflections by $x_{11}$ and $x_{21}$,respectively, we write

$$x_{11} = \alpha_{11}P_1 \qquad x_{21} = \alpha_{21}P_1 \qquad \text{...(41)}$$

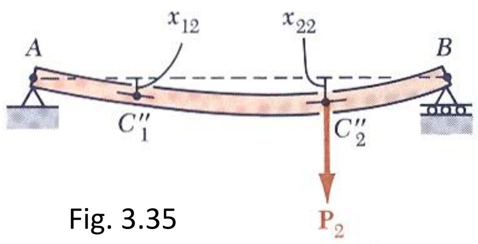


Fig. 3.35

where α11 and α21 are constants called influence coefficients. These constants represent the deflections of $C_1$ and $C_2$, respectively, when a unit load is applied at $C_1$ and are characteristics of the beam AB.

Let us now assume that only $P_2$ is applied to the beam (Fig. 3.35).

Denoting by $x_{12}$ and $x_{22}$, respectively, the resulting deflections of $C_1$ and$C_2$, we write

$$x_{12} = \alpha_{12}P_2 \qquad x_{22} = \alpha_{22}P_2 \qquad \text{...(42)}$$

where $\alpha_{12}$ and $\alpha_{22}$ are the influence coefficients representing the deflections of $C_1$ and $C_2$, respectively, when a unit load is applied at $C_2$. Applying the principle of superposition, we express the deflections $x_1$ and $x_2$ of $C_1$ and $C_2$ when both loads are applied (Fig. 3.33) as

$$x_1 = x_{11} + x_{12} = \alpha_{11}P_1 + \alpha_{12}P_2 \qquad \text{...(43)}$$

$$x_2 = x_{21} + x_{22} = \alpha_{21}P_1 + \alpha_{22}P_2 \qquad \text{...(44)}$$

To compute the work done by P1 and P2, and thus the strain energy of the beam, we shall find it convenient to assume that P1 is first applied slowly at C1 (Fig. 3.36a).

Recalling the first of Eqs. (3.41), we expressthe work of $P_1$ as

$$\tfrac{1}{2}P_1x_{11} = \tfrac{1}{2}P_1(\alpha_{11}P_1) = \tfrac{1}{2}\alpha_{11}P_1^2 \qquad \text{...(45)}$$

and note that $P_2$ does no work while $C_2$ moves through $x_{21}$, since it has not yet been applied to the beam.

Now we slowly apply $P_2$ at $C_2$ (Fig. 2.36b); recalling the second of Eqs. (42), we express the work of $P_2$ as

$$\tfrac{1}{2}P_2x_{22} = \tfrac{1}{2}P_2(\alpha_{22}P_2) = \tfrac{1}{2}\alpha_{22}P_2^2 \qquad ...(46)$$

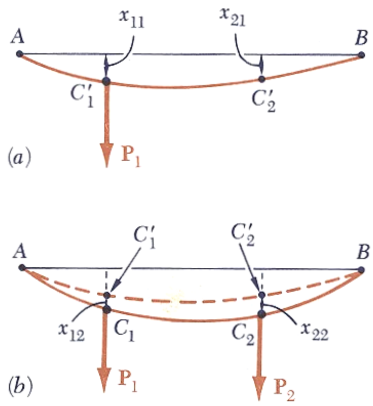


Fig 3.36

But, as $P_2$ is slowly applied at $C_2$, the point of application of $P_1$ moves through from $x_{12}$, from $C_1'$ to $C_1$, and the load $P_1$ does work. Since $P_1$ is fully applied during this displacement (Fig, 3.37), its work is equal to $P_1x_{12}$ or, recalling the first of Eqs. (42),

$$P_1x_{12} = P_1(\alpha_{12}P_2) = \alpha_{12}P_1P_2 \qquad ...(47)$$

Adding the expressions obtained in (45), (46), and (47), we express the strain energy of the beam under the loads $P_1$ and $P_2$ as

$$U = \tfrac{1}{2}(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2) \qquad ...(48)$$

If the load $P_2$ had first been applied to the beam (Fig. 3.38a), and then the load $P_1$ (Fig. 3.38b), the work done by each load would have been as shown in Fig. 3.39.

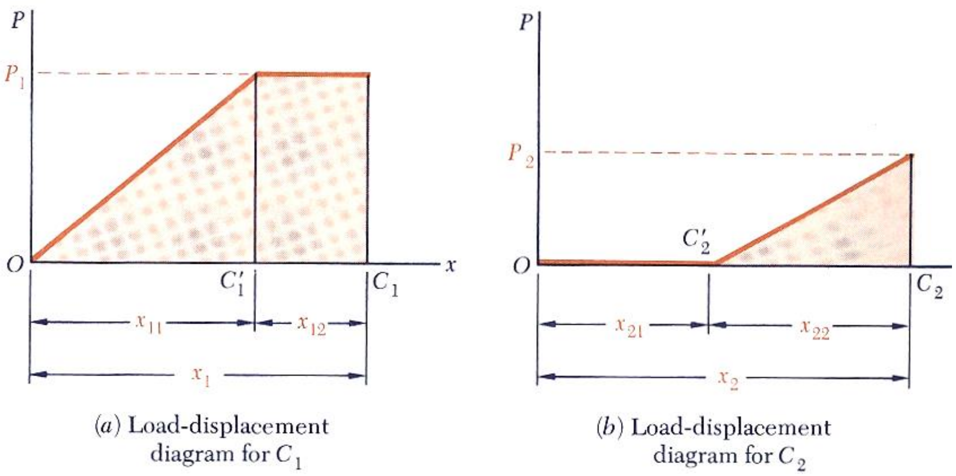



Fig. 3.37 2.37

Calculations similar to those we have just carried out would lead to the following alternative expression for the strain energy of the beam:

$$U = \tfrac{1}{2}(\alpha_{22}P_2^2 + 2\alpha_{21}P_2P_1 + \alpha_{11}P_1^2) \qquad \text{...(49)}$$

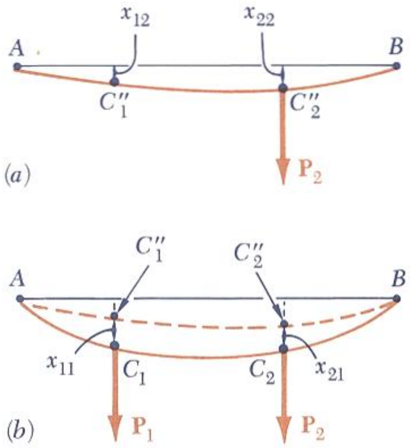


Fig. 3.38

Equating the right-hand members of Eqs. (48) and (49), we find that $\alpha_{12} = \alpha_{21}$, and thus conclude that the deflection produced at $C_1$ by a unit load applied at $C_2$ is equal to the deflection produced at $C_2$ by a unit load applied at $C_1$. This is known as Maxwell's reciprocal theorem, after the British physicist James Clerk Maxwell (1831-1879).

While we are now able to express the strain energy U of a structure subjected to several loads as a function of these loads, we cannot use the method of Sec. 3.9 to determine the deflection of such a structure. Indeed, computing the strain energy U by integrating the strain-energy density u over the structure and substituting the expression obtained into (48) would yield only one equation, which clearly could not be solved for the various coefficients α.

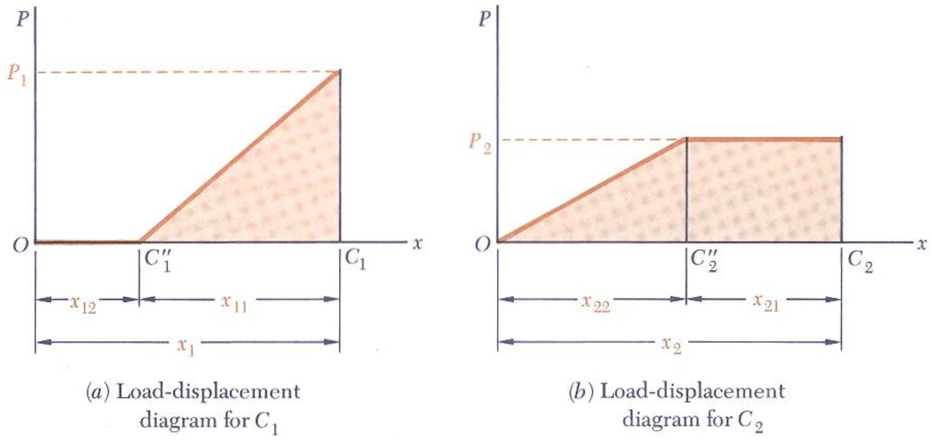


Fig. 3.39

## 3.11. CASTIGLIANO'S THEOREM

We recall the expression obtained in the preceding section for the strain energy of an elastic structure subjected to two loads $P_1$ and $P_2$:

$$U = \tfrac{1}{2}(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2) \qquad \text{...(50)}$$

where $\alpha_{11}$, $\alpha_{12}$ and $\alpha_{22}$ are the influence coefficients associated with the points of application $C_1$ and $C_2$ of the two loads. Differentiating both members of Eq. (48) with respect to $P_1$ and recalling Eq. (43), we write

$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1 \qquad \text{...(51)}$$

Differentiating both members of Eq. (50) with respect to $P_2$, recalling Eq. (44), and keeping in mind that $\alpha_{12} = \alpha_{21}$, we have

$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2 \qquad \text{...(52)}$$

More generally, if an elastic structure is subjected to n loads $P_1, P_2, \ldots, P_n$ the deflection of the point of application of $P_f$, measured alongthe line of action of $P_f$, may be expressed as the partial derivative of the strain energy of the structure with respect to the load P. We write

$$x_j = \frac{\partial U}{\partial P_j} \qquad \text{...(53)}$$

This is Castigliano's theorem, named after the Italian engineer AlbertoCastigliano (1847-1884) who first stated it.

Recalling that the work of a couple M is $\frac{1}{2}M\theta$, where $\theta$ is the angle of rotation at the point where the couple is slowly applied, we note that Castigliano's theorem may be used to determine the slope of a beam at the point of application of a couple $M_j$. We have

$$\theta_j = \frac{\partial U}{\partial M_j} \qquad \text{...(54)}$$

Similarly, the angle of twist $\phi_j$ in a section of a shaft where a torque $T_j$ is slowly applied is obtained by differentiating the strain energy of the shaft with respect to $T_j$ :

$$\phi_j = \frac{\partial U}{\partial T_j} \qquad \text{...(55)}$$

### 3.11. DEFLECTIONS BY CASTIGLIANO'S THEOREM

We saw in the preceding section that the deflection $x_j$ of a structure at the point of application of a load P may be determined by computing the partial derivative $\partial U/\partial P_j$ of the strain energy U of the structure. As we recall from Secs. 3.4 and 3.5, the strain energy U is obtained by integrating or summing over the structure the strain energy of each element of the structure. We shall find that the calculation by Castigliano's theorem of the deflection $x_j$ is simplified if the differentiation with respect to the load $P_j$ is carried out before the integration or summation.

In the case of a beam, for example, we recall from Sec. 3.4 that

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad \text{...(56)}$$

and determine the deflection $x_j$ of the point of application of the load $P_j$ by writing

$$x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_j} dx \qquad \text{...(57)}$$

In the case of a truss consisting of n uniform members of length $L_i$, cross-sectional area $A_i$, and internal force $F_i$, we recall Eq. (14) and express the strain energy U of the truss as

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E} \qquad \text{...(58)}$$

The deflection $x_j$ of the point of application of the load $P_j$ is obtained by differentiating with respect to each term of sum. We write

$$x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j} \qquad \text{...(59)}$$

**EXERCISES**

Strain Energy & Strain energy density

**1.** Determine the modulus of resilience for each of the following metals:

(a) Stainless steel (cold-rolled): E = 190 GPa, $s_y$ = 520 MPa

(b) Stainless steel (annealed): E = 190 GPa, $s_y$ = 260 MPa

(c) Malleable cast iron: E = 165 GPa, $s_y$ = 230 MPa

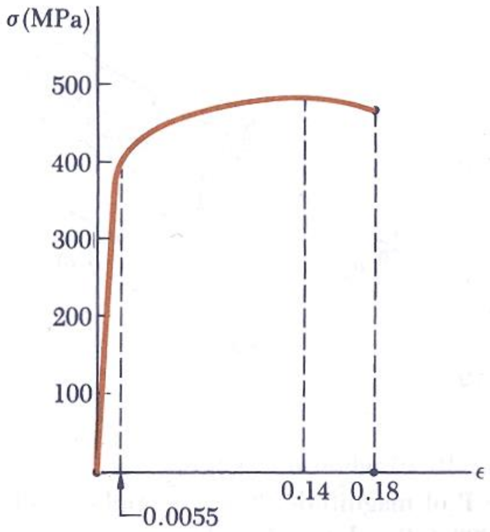



Fig. 1

**2.** The stress-strain diagram shown in Fig. 1 has been drawn from data obtained during the tensile test of an aluminum alloy. Using E = 72 GPa, (a) determ the modulus of resilience of the alloy, (b) determine by approximate means the modulus of toughness of the alloy.

**3.** Using E = 75 GPa, determine (a) the strain energy of the aluminum rod ABC when a = 1.2 m, (b) the corresponding strain-energy density in portions AB and BC of the rod. (Refer Fig. 2 )

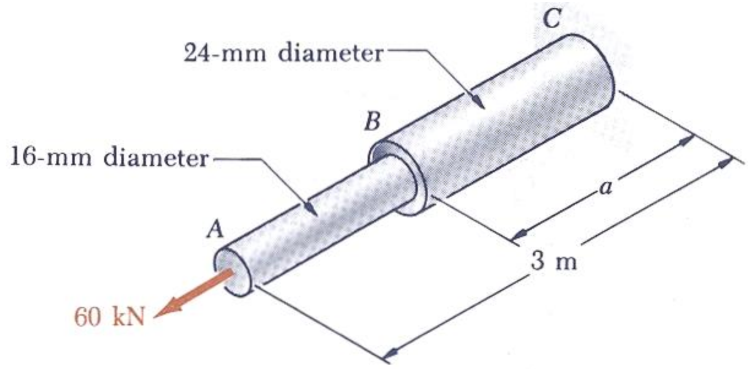



Fig. 2

**4.** In the truss shown in Fig. 3 , all members are made of the same material and have the uniform cross-sectional areas indicated. Determine the strain energy of the truss when the load P is applied.

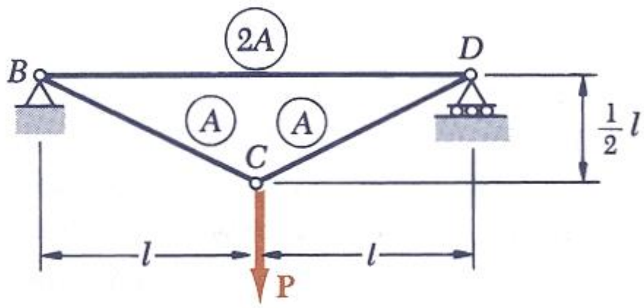



Fig. 3

**5.** Assuming that the prismatic beam AB in Fig.4 has a rectangular cross section, show that for the given loading the maximum value of the strain-energy density in the beam is

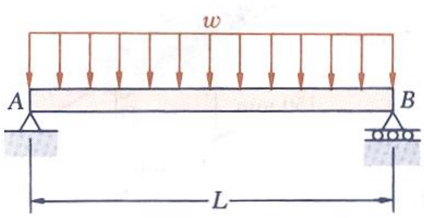



Fig. 4

$$u_m = \frac{45}{8}\frac{U}{V}$$

where U is the strain energy of the beam and V is its volume.

**6.** Two solid shafts are connected by the gears shown in Fig. 5. Using G = 77 GPa , determine the strain energy of each shaft when a 1.1 kN- m torque T is applied at D.

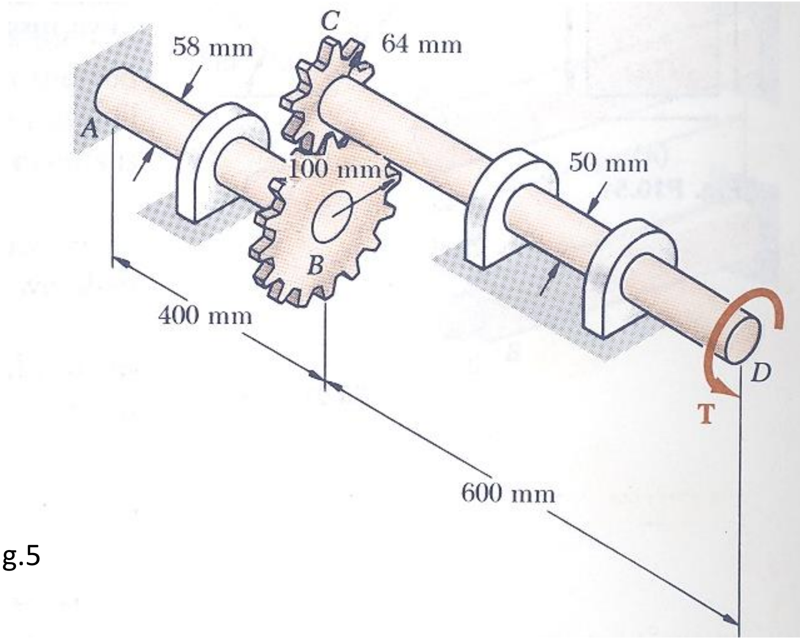



Fig.5

**7.** Determine the strain energy of the prismatic beam AB shown in Fig. 6, taking into account the effect of both normal and shearing stresses.

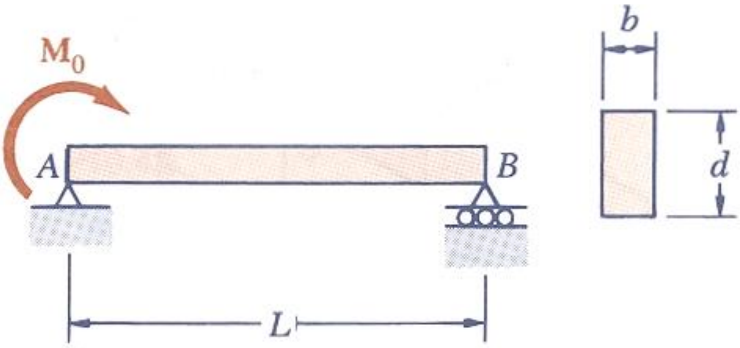



Fig. 6

**8.** An aluminum tube having the cross section shown in Fig.7 is struck squarely in its midsection by a 6-kg block moving horizontally with a speed of 2 m/s. Using E = 70 GPa, determine (a) the equivalent static load, (b) the maximum stress in the beam, (c) the maximum deflection at the midpoint C of the beam.

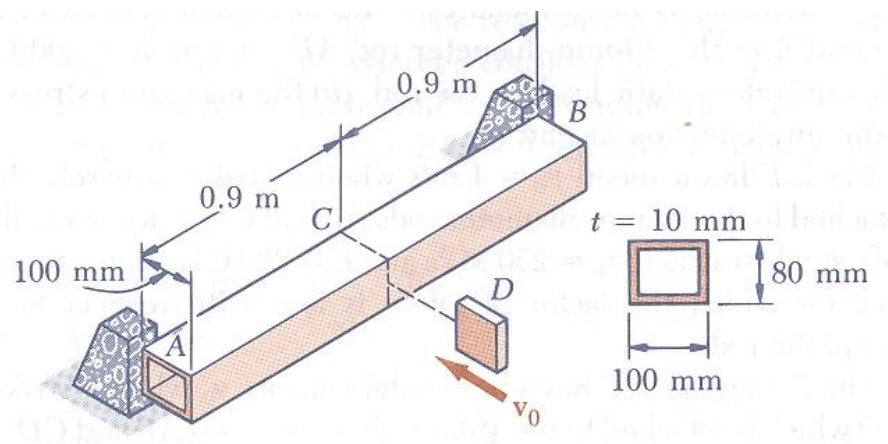



Fig. 7

9. Using the method of work and energy, determine the deflection at point D caused by the load P. (Refer Fig. 8)

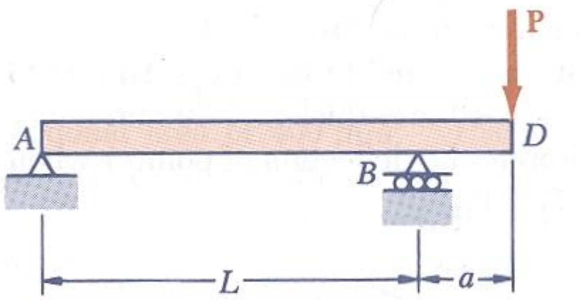



Fig. 8

**10.** The 20-mm-diameter steel rod BC is attached to the lever AB and to the fixed support C shown in Fig.9. The uniform steel lever is 10 mm thick and 30 mm deep. Using the method of work and energy, determine the length L of rod BC for which the deflection of point A is 40 mm. Use E = 200 GPa and G = 77 GPa.

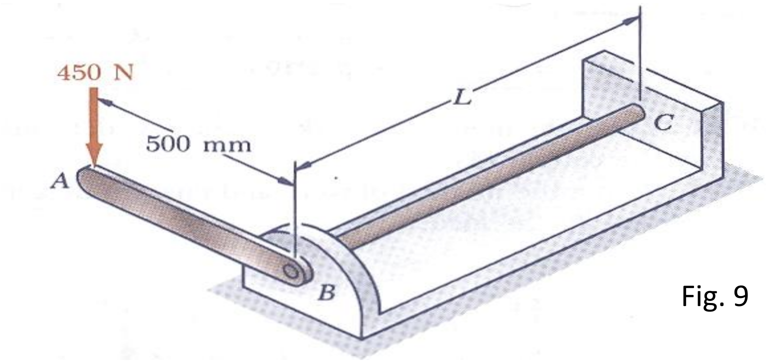
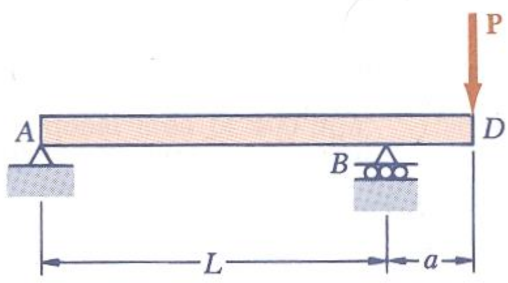



Fig. 9

<u>work and energy method</u>

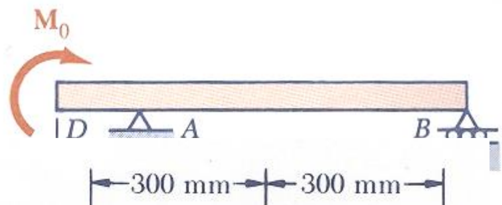



**11.** Using the method of work and energy, determine the deflection at point D caused by the load P.

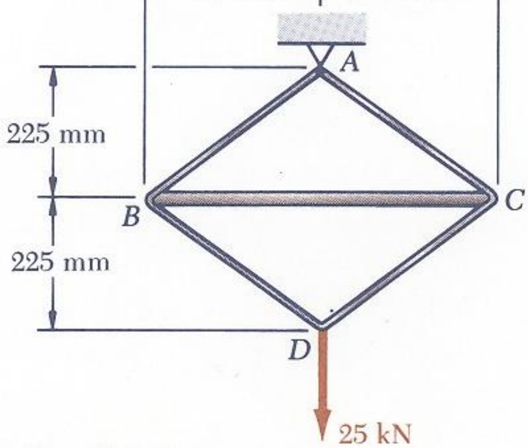
**Ans.[** $pa^2(a+L)/3EI \downarrow$]

**12.** Using the method of work and energy, determine the slope at point D caused by the couple $M_0$.

**[Ans.** $M_o(L+3a)/3EI$ **]**




**13.** In the assembly shown, member BC is a solid steel rod of 25-mm diameter and all other members are made of 12.5



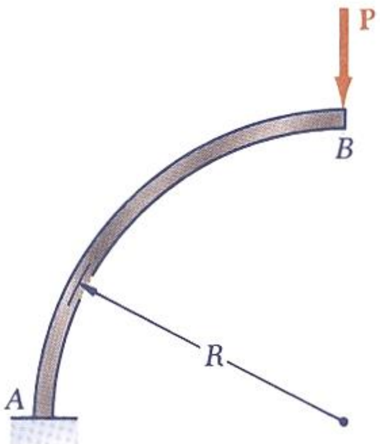
mm diameter steel rods. Using F 200 GPa,  determine the deflection of point D caused by the 25-kN load.

**[Ans. 1.333 mm ↓ ]**

<u>Castigliano's theorem</u>

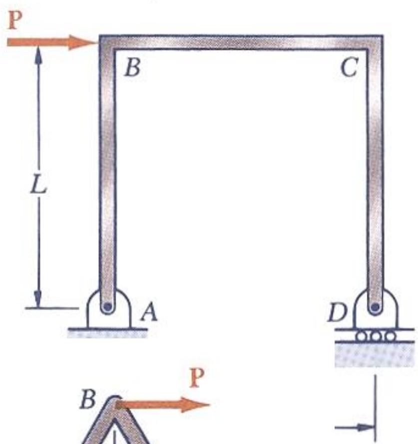
**14.** For the uniform rod and loading shown and using Castigliano's theorem, determine (a) the horizontal deflection of point B, (b) the vertical deflection of point B.

**[Ans. a)** $PR^3/2EI \rightarrow \quad b)\, \pi PR^3/4EI \downarrow$ **]**



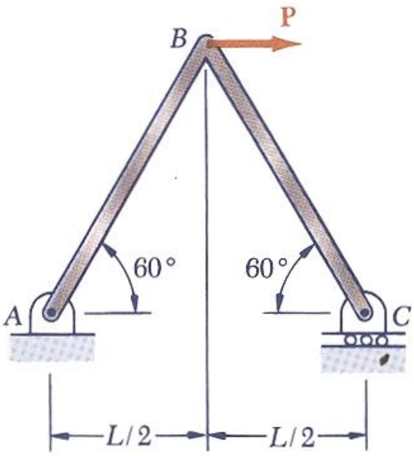

**15.** Three rods, each of the same flexural rigidity El, are welded to form the frame ABCD. For the loading shown, determine (a) the deflection of point D, (b) the angle formed by the frame at point D and the vertical.

**[Ans. (a)** $5PL^3/6EI \rightarrow; (b)\, PL^2/6EI$ **]**




**16.** Two rods of the same flexural rigidity El are welded at B. For the loading shown, determine (a) the horizontal deflection of point B, (b) the vertical deflection of point B.

**[ Ans. Find yourself ]**



## REFERENCES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2. Gere, James M., Mechanics of materials, 6th ed., 2003.